

The lattice point discrepancy of a body of revolution: Improving the lower bound by Soundararajan's method

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Abstract. For a convex body \mathcal{B} in \mathbb{R}^3 which is invariant under rotations around one coordinate axis and has a smooth boundary of bounded nonzero curvature, the *lattice point discrepancy* $P_{\mathcal{B}}(t)$ (number of integer points minus volume) of a linearly dilated copy $\sqrt{t}\mathcal{B}$ is estimated from below. On the basis of a recent method of K. Soundararajan [16] an Ω -bound is obtained that improves upon all earlier results of this kind.

1. Introduction. We consider a compact convex body \mathcal{B} in \mathbb{R}^3 which contains the origin as an inner point and assume that its boundary $\partial\mathcal{B}$ is a C^∞ surface⁽¹⁾ with bounded nonzero Gaussian curvature throughout. For a large real parameter t , we consider a linearly dilated copy $\sqrt{t}\mathcal{B}$ of \mathcal{B} , and in particular its *lattice point discrepancy*

$$P_{\mathcal{B}}(t) := \# \left(\sqrt{t}\mathcal{B} \cap \mathbb{Z}^3 \right) - \text{vol}(\mathcal{B})t^{3/2}. \quad (1.1)$$

There is a rich and very classic theory dealing with the estimation of such quantities $P_{\mathcal{B}}(t)$, both in arbitrary dimensions and for very special cases. An enlightening survey can be found in E. Krätzel's monographs [8] and [9] which have to be supplemented by M. Huxley's book [7] where he exposed his breakthrough in planar lattice point theory (*Discrete Hardy-Littlewood method*).

For our specific setting stated above, the sharpest results read

$$P_{\mathcal{B}}(t) = O \left(t^{63/86+\varepsilon} \right) \quad (1.2)$$

and⁽²⁾

$$P_{\mathcal{B}}(t) = \Omega_- \left(t^{1/2}(\log t)^{1/3} \right). \quad (1.3)$$

These are due to W. Müller [14] (who improved earlier results by E. Hlawka [5] and Krätzel and Nowak [10], [11]), and the second named author [15], respectively.

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(1) This assumption will be made a bit more precise at the end of section 2.

(2) For the definitions of the different Ω -symbols, cf. Krätzel [8], p. 14.

In recent years, it has been noted that sharper estimates are true for a body \mathcal{B} which is invariant under rotations around one of the coordinate axes. In this case,

$$P_{\mathcal{B}}(t) = O\left(t^{11/16}\right), \quad (1.4)$$

according to F. Chamizo [1], and⁽³⁾

$$P_{\mathcal{B}}(t) = \Omega_{-}\left(t^{1/2}(\log t)^{1/3}(\log_2 t)^{\frac{1}{3}\log 2}\exp(-c\sqrt{\log_3 t})\right), \quad c > 0, \quad (1.5)$$

as was shown by the first named author [12], on the basis of a deep and fairly general method of J.L. Hafner [3].

Quite recently, K. Soundararajan [16] exploited a brilliant new idea to obtain sharper Ω -estimates in the classic circle and divisor problems. In the present note we will apply this ingenious new approach to improve⁽⁴⁾ the lower bound of (1.5).

Theorem. *Let \mathcal{B} be a compact convex body in \mathbb{R}^3 which is invariant under rotations around one of the coordinate axes and contains $(0, 0, 0)$ as an inner point. Assume that its boundary $\partial\mathcal{B}$ is of class C^∞ and has bounded nonzero Gaussian curvature throughout. Then*

$$P_{\mathcal{B}}(t) = \Omega_{-}\left(t^{1/2}(\log t)^{1/3}(\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 t)^{-2/3}\right).$$

We remark parenthetically that still much sharper estimates are known for the special case that \mathcal{B} is the unit ball \mathcal{B}_0 in \mathbb{R}^3 (*sphere problem*). In fact, Heath-Brown [4] obtained⁽⁵⁾

$$P_{\mathcal{B}_0}(t) = O\left(t^{21/32+\varepsilon}\right), \quad (1.6)$$

thereby improving a result of Chamizo and Iwaniec [2] and earlier classic work of I.M. Vinogradov [20]. In the other direction, K.-M. Tsang [19] showed that

$$P_{\mathcal{B}_0}(t) = \Omega_{\pm}\left(t^{1/2}(\log t)^{1/2}\right), \quad (1.7)$$

the Ω_{-} -part of this result being much older and actually due to G. Szegő [17].

(3) By \log_j , $j = 2, 3, \dots$, we denote throughout the j -fold iterated logarithm.

(4) Note that $\frac{1}{3}\log 2 = 0.2310\dots$ while $\frac{2}{3}(\sqrt{2}-1) = 0.2761\dots$

(5) It is instructive to compare the numerical values of the exponents in (1.2), (1.4), and (1.6): $\frac{63}{86} = 0.7325\dots$, $\frac{11}{16} = 0.6875$, $\frac{21}{32} = 0.65625$.

2. Preliminaries.

Soundararajan's Lemma [16]. *Let $(f(n))_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ be sequences of non-negative real numbers, $(\lambda_n)_{n=1}^{\infty}$ non-decreasing, and $\sum_{n=1}^{\infty} f(n) < \infty$. Let $L \geq 2$ be an integer and Λ a positive real parameter. Suppose further that \mathcal{M} is a finite set of positive integers, such that $\{\lambda_m : m \in \mathcal{M}\} \subset [\frac{1}{2}\Lambda, \frac{3}{2}\Lambda]$. Then, for any real $T \geq 2$, there exists some $t \in [\frac{1}{2}T, (6L)^{|\mathcal{M}|+1}T]$ with*

$$\sum_{n=1}^{\infty} f(n) \cos(2\pi \lambda_n t) \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{1}{L-1} \sum_{n: \lambda_n \leq 2\Lambda} f(n) - \frac{2}{\pi^2 T \Lambda} \sum_{n=1}^{\infty} f(n).$$

We further notice some important properties of the *tac function* H of a convex body \mathcal{B} with the properties stated above. This is defined by

$$H(\mathbf{w}) = \max_{\mathbf{x} \in \mathcal{B}} (\mathbf{x} \cdot \mathbf{w}) \quad (\mathbf{w} \in \mathbb{R}^3)$$

where \cdot denotes the standard inner product. From this the following facts are evident:

- (i) H is positive and homogeneous of degree 1.
- (ii) There exist constants $c_2 > c_1 > 0$, depending on \mathcal{B} , such that for all $\mathbf{w} \in \mathbb{R}^3$

$$c_1 \|\mathbf{w}\| \leq H(\mathbf{w}) \leq c_2 \|\mathbf{w}\|, \quad (2.1)$$

where $\|\cdot\|$ stands for the Euclidean norm throughout.

- (iii) If \mathcal{B} is invariant with respect to rotations around the third coordinate axis (say), then so is H , i.e., for all $(w_1, w_2, w_3) \in \mathbb{R}^3$,

$$H(w_1, w_2, w_3) = H(\sqrt{w_1^2 + w_2^2}, 0, w_3). \quad (2.2)$$

It seems appropriate to say a bit more about the smoothness condition that $\partial\mathcal{B}$ be of class C^∞ . Properly speaking, this is supposed to mean that for every point of $\partial\mathcal{B}$ there exists a neighbourhood in which the corresponding portion of $\partial\mathcal{B}$ has a regular⁽⁶⁾ parametrization $\mathbf{x} = \mathbf{x}(u_1, u_2)$ whose components are all of class C^∞ . However, as has been neatly worked out in W. Müller [13], Lemmas 1 and 2, this local property implies that the *spherical map*, which sends every point of the unit sphere into that point of $\partial\mathcal{B}$ where the outward normal has the same direction, is globally one-one and C^∞ . Under these latter conditions, Hlawka's asymptotic formulas for the Fourier transform of the indicator function of \mathcal{B} had been established [5], [6]. These in turn have been used in [15], upon which our present analysis will be based.

For the case that \mathcal{B} is a body of revolution (with respect to the x_3 -axis, say), the conditions of our Theorem can be stated in a more concise form. It suffices to assume that

$$\partial\mathcal{B} = \{\mathbf{x} = (x_1, x_2, x_3) = (\rho(\theta) \sin(\theta) \cos(\phi), \rho(\theta) \sin(\theta) \sin(\phi), \rho(\theta) \cos(\theta)) : 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi\},$$

(6) I.e., $\frac{\partial \mathbf{x}}{\partial u_1}, \frac{\partial \mathbf{x}}{\partial u_2}$ are linearly independent.

where $\rho : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is an even function, periodic with period 2π and everywhere of class C^∞ , which satisfies throughout

$$\rho \rho'' - 2\rho'^2 - \rho^2 \neq 0. \quad (2.3)$$

In fact, the Gaussian curvature κ_3 of this surface $\partial\mathcal{B}$ is readily computed as

$$\kappa_3(\theta) = \frac{\frac{dx_3}{d\theta}}{\rho(\theta) \sin(\theta)} \frac{\rho(\theta) \rho''(\theta) - 2\rho'^2(\theta) - \rho^2(\theta)}{(\rho^2(\theta) + \rho'^2(\theta))^2}.$$

We may imagine $\partial\mathcal{B}$ to be generated by rotation of the *meridian*

$$\{(x_1, x_3) = (\rho(\theta) \sin(\theta), \rho(\theta) \cos(\theta)) : 0 \leq \theta \leq \pi\}$$

around the x_3 -axis. The curvature κ_2 of the latter satisfies

$$|\kappa_2(\theta)| = \frac{|\rho(\theta) \rho''(\theta) - 2\rho'^2(\theta) - \rho^2(\theta)|}{(\rho^2(\theta) + \rho'^2(\theta))^{3/2}}.$$

Therefore, (2.3) guarantees the nonvanishing of κ_2 , and also that of κ_3 , since by geometric evidence $\frac{dx_3}{d\theta} > 0$ for $0 < \theta < \pi$.

3. Proof of the Theorem. For real $t > 0$, we put

$$X = X(t) = (\log t)^{-1}, \quad k = k(t) = t^2 \log t, \quad (3.1)$$

then the *Borel mean-value* of the lattice rest $P_{\mathcal{B}}$ is defined as

$$B(t) := \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k P_{\mathcal{B}}(Xu) du. \quad (3.2)$$

We start from formula (13) in [15]: For large t , and arbitrary $\varepsilon > 0$,

$$B(t) = -\frac{1}{2\pi} t S(t) + O\left(t^{3/8+\varepsilon}\right), \quad (3.3)$$

where

$$S(t) := \sum_{0 < \|\mathbf{m}\| \leq t^{\varepsilon_0} X^{-1/2}} \frac{\alpha(\mathbf{m})}{\|\mathbf{m}\|^2} \exp\left(-\frac{1}{2}\pi^2 X H(\mathbf{m})^2\right) \cos(2\pi H(\mathbf{m})t). \quad (3.4)$$

Here $\varepsilon_0 > 0$ is a sufficiently small constant, $\mathbf{m} = (m_1, m_2, m_3)$ denotes elements of \mathbb{Z}^3 throughout, and the coefficients $\alpha(\mathbf{m})$ are positive reals bounded both from above and away from 0. By (2.2), we can rewrite this last formula as

$$S(t) = \sum_{0 < \ell + m_3^2 \leq t^{2\varepsilon_0} \log t} \frac{g(\ell, m_3)}{\ell + m_3^2} \exp\left(-\frac{1}{2}\pi^2 X H(\sqrt{\ell}, 0, m_3)^2\right) \cos(2\pi H(\sqrt{\ell}, 0, m_3)t),$$

with

$$g(\ell, m_3) := \sum_{\substack{(m_1, m_2) \in \mathbb{Z}^2: \\ m_1^2 + m_2^2 = \ell}} \alpha(m_1, m_2, m_3) \asymp r(\ell), \quad (3.5)$$

$r(\ell)$ the number of ways to write $\ell \in \mathbb{N}$ as a sum of two squares of integers.

In order to apply Soundararajan's Lemma, we consider a one-one map \mathbf{q} of \mathbb{N}_* onto $\mathbb{N} \times \mathbb{Z} \setminus \{(0, 0)\}$, $n \mapsto \mathbf{q}(n) = (\ell, m_3)$ such that the sequence $(\lambda_n)_{n=1}^\infty$ defined by

$$\lambda_n := H(\sqrt{\ell}, 0, m_3) \Big|_{(\ell, m_3) = \mathbf{q}(n)} \quad (3.6)$$

is non-decreasing⁽⁷⁾. Putting further

$$f(n) := \frac{g(\ell, m_3)}{\ell + m_3^2} \exp(-\frac{1}{2}\pi^2 X H(\sqrt{\ell}, 0, m_3)^2) \Big|_{(\ell, m_3) = \mathbf{q}(n)} \quad (3.7)$$

if $\ell + m_3^2 \leq t^{2\varepsilon_0} \log t$, and $f(n) = 0$ else, we obtain in fact

$$S(t) = \sum_{n=1}^{\infty} f(n) \cos(2\pi \lambda_n t),$$

and are thus prepared to apply Soundararajan's Lemma. For $T \geq 40$ a large real parameter, we put $L = \lceil (\log_2 T)^{20} \rceil$ and assume that the set \mathcal{M} will be chosen such that

$$(6L)^{|\mathcal{M}|+1} \leq T. \quad (*)$$

Then, by Soundararajan's Lemma, there exists a value $t \in [\frac{1}{2}T, T^2]$ for which

$$S(t) \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{1}{L-1} \sum_{n: \lambda_n \leq 2\Lambda} f(n) - \frac{2}{\pi^2 T \Lambda} \sum_{n=1}^{\infty} f(n), \quad (3.8)$$

where $\Lambda > 0$ is a parameter remaining to be determined.

By homogeneity of the tac-function H , there exist positive constants $a_2 > a_1 > 0$ and $a_3 > a_4 > 0$ depending on \mathcal{B} such that the two-dimensional interval $[a_1, a_2] \times [a_3, a_4]$ in the (w_1, w_3) -plane, say, lies between the two curves $H(w_1, 0, w_3) = \frac{1}{2}$ and $H(w_1, 0, w_3) = \frac{3}{2}$. Consequently, for integers $\ell > 0$ and m_3 , the condition $(\sqrt{\ell}, m_3) \in [a_1\Lambda, a_2\Lambda] \times [a_3\Lambda, a_4\Lambda]$ always implies that $H(\sqrt{\ell}, 0, m_3) \in [\frac{1}{2}\Lambda, \frac{3}{2}\Lambda]$.

Let us denote by \mathbb{A}_1 the set of positive integers whose prime divisors are all congruent to 1 mod 4, and by $\omega(\ell)$ the number of prime divisors of $\ell \in \mathbb{N}_*$.

(7) In other words: We arrange the elements (ℓ, m_3) of $\mathbb{N} \times \mathbb{Z} \setminus \{(0, 0)\}$ according to the size of the values $H(\sqrt{\ell}, 0, m_3)$.

Then we define

$$\widehat{\mathcal{M}} = \{(\ell, m_3) \in \mathbb{N}_*^2 : a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, a_3 \Lambda \leq m_3 \leq a_4 \Lambda, \ell \in \mathbb{A}_1, \omega(\ell) = [\beta \log_2 \Lambda]\},$$

where $\beta > 0$ is a coefficient whose optimal choice ultimately will be $\beta = \sqrt{2}$.

Let \mathcal{M} be the preimage of $\widehat{\mathcal{M}}$ under the map \mathbf{q} . By construction, $\{\lambda_m : m \in \mathcal{M}\} \subset [\frac{1}{2}\Lambda, \frac{3}{2}\Lambda]$, as required in Soundararajan's Lemma.

By (3.5) and (3.7),

$$\begin{aligned} \sum_{m \in \mathcal{M}} f(m) &\gg \frac{1}{\Lambda^2} \sum_{a_3 \Lambda \leq m_3 \leq a_4 \Lambda} \sum_{\substack{a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \\ \ell \in \mathbb{A}_1, \omega(\ell) = [\beta \log_2 \Lambda]}} r(\ell) \\ &\gg \frac{1}{\Lambda} \sum_{a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \ell \in \mathbb{A}_1, \omega(\ell) = [\beta \log_2 \Lambda]} r(\ell), \end{aligned} \tag{3.9}$$

where we have been assuming for the moment that

$$XH(\sqrt{\ell}, 0, m_3)^2 \ll 1 \tag{**}$$

for the values of ℓ and m_3 involved.

Furthermore, $r(\ell) \geq 2^{\omega(\ell)}$ for $\ell \in \mathbb{A}_1$, and the cardinality of

$$\mathcal{S}_{\Lambda, K} := \{\ell \in \mathbb{N}_* : a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \ell \in \mathbb{A}_1, \omega(\ell) = K\}$$

is readily estimated after the example of Tenenbaum [18], section II.6. One may start from the observation that, for $\Re(s) > 1$, $z \in \mathbb{C}$ arbitrary,

$$\sum_{n \in \mathbb{A}_1} z^{\omega(n)} n^{-s} = \prod_{p \equiv 1 \pmod{4}} \left(1 + \frac{z}{p^s - 1}\right) = (\zeta_{\mathbb{Q}(i)}(s))^{z/2} G(s; z),$$

where $\zeta_{\mathbb{Q}(i)}$ is the Dedekind zeta-function of the Gaussian field, and $G(s; z)$ is holomorphic and bounded in every half-plane $\Re(s) \geq \sigma_0 > \frac{1}{2}$. It follows⁽⁸⁾ that, as long as $K \ll \log_2 \Lambda$,

$$|\mathcal{S}_{\Lambda, K}| \asymp \frac{\Lambda^2}{\log \Lambda} \frac{(\frac{1}{2} \log_2 \Lambda)^{K-1}}{(K-1)!}.$$

With Stirling's formula in the shape $(K-1)! \asymp K^{K-1/2} e^{-K}$ and the choice $K = [\beta \log_2 \Lambda]$, this gives

$$|\mathcal{S}_{\Lambda, K}| \asymp \frac{\Lambda^2}{\sqrt{\log_2 \Lambda}} (\log \Lambda)^{\beta-1-\beta \log(2\beta)},$$

⁽⁸⁾ This has been noticed already by Soundararajan [16], f. (3.7). The authors intend to carry out the details for the case of a general number field \mathbb{K} in a forthcoming article.

and thus

$$|\mathcal{M}| = |\widehat{\mathcal{M}}| \asymp \frac{\Lambda^3}{\sqrt{\log_2 \Lambda}} (\log \Lambda)^{\beta-1-\beta \log(2\beta)}, \quad (3.10)$$

Therefore, recalling (3.9) and the fact that $r(\ell) \geq 2^{\omega(\ell)}$ for $\ell \in \mathbb{A}_1$, we obtain

$$\sum_{m \in \mathcal{M}} f(m) \gg \frac{\Lambda}{\sqrt{\log_2 \Lambda}} (\log \Lambda)^{\beta-1-\beta \log \beta}. \quad (3.11)$$

We now have to choose Λ such that $(*)$ is satisfied. This is done optimally as

$$\Lambda = c_0 (\log T)^{1/3} (\log_2 T)^{\frac{1}{3}(1-\beta+\beta \log(2\beta))} (\log_3 T)^{-1/6}, \quad (3.12)$$

where c_0 is an appropriate small constant. As a consequence, $(**)$ is verified, since $X \ll (\log T)^{-1}$ and $H(\sqrt{\ell}, 0, m_3) \ll \Lambda$ for the values of ℓ and m_3 involved. Furthermore, $\log \Lambda \asymp \log_2 T$ and $\log_2 \Lambda \asymp \log_3 T$, thus ultimately

$$\sum_{m \in \mathcal{M}} f(m) \gg (\log T)^{1/3} (\log_2 T)^{\frac{2}{3}(\beta-1-\beta \log \beta) + \frac{1}{3}\beta \log 2} (\log_3 T)^{-2/3}.$$

Here the second exponent is maximized for $\beta = \sqrt{2}$, and we finally obtain

$$\sum_{m \in \mathcal{M}} f(m) \gg (\log T)^{1/3} (\log_2 T)^{\frac{2}{3}(\sqrt{2}-1)} (\log_3 T)^{-2/3}. \quad (3.13)$$

It remains to show that the two other terms on the right hand side of (3.8) are small.

In fact,

$$\begin{aligned} \sum_{n: \lambda_n \leq 2\Lambda} f(n) &\ll \sum_{0 < H(\sqrt{\ell}, 0, m_3) \leq 2\Lambda} \frac{r(\ell)}{\ell + m_3^2} = \sum_{0 < H(\mathbf{m}) \leq 2\Lambda} \|\mathbf{m}\|^{-2} \leq \\ &\leq \sum_{0 < c_1 \|\mathbf{m}\| \leq 2\Lambda} \|\mathbf{m}\|^{-2} = \sum_{1 \leq n \leq (4/c_1^2)\Lambda^2} \frac{r_3(n)}{n} = \int_{1-}^{(4/c_1^2)\Lambda^2} \frac{1}{u} d \left(\sum_{1 \leq n \leq u} r_3(n) \right) \ll \Lambda, \end{aligned}$$

using integration by parts of Stieltjes integrals and the well-known bound $\sum_{1 \leq n \leq u} r_3(n) \ll u^{3/2}$.

After division by $L - 1$, which by construction is $\asymp (\log_2 T)^{20}$, this is small compared to the right-hand side of (3.13).

Similarly (for the value of $t \in [\frac{1}{2}T, T^2]$ specified by Soundararajan's Lemma),

$$\begin{aligned} \frac{2}{\pi^2 T \Lambda} \sum_{n=1}^{\infty} f(n) &\ll \frac{1}{T \Lambda} \sum_{0 < \|\mathbf{m}\| \leq t^{\varepsilon_0} / \sqrt{X}} \|\mathbf{m}\|^{-2} = \\ &= \frac{1}{T \Lambda} \int_{1-}^{t^{2\varepsilon_0} \log t} \frac{1}{u} d \left(\sum_{1 \leq n \leq u} r_3(n) \right) \ll T^{3\varepsilon_0-1}. \end{aligned}$$

Combining the last two bounds with (3.8) and (3.3), we conclude that for arbitrary $T \geq 40$, there exists a value $t \in [\frac{1}{2}T, T^2]$ with

$$-B(t) \gg t(\log t)^{1/3}(\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 t)^{-2/3}. \quad (3.14)$$

Let us assume that, with some constants C and $\varepsilon_1 > 0$, and for all $u > 0$,

$$-P_B(u) \leq C + \varepsilon_1 u^{1/2} \mathcal{L}(u),$$

where

$$\mathcal{L}(u) := (\log u)^{1/3}(\log_2 u)^{\frac{2}{3}(\sqrt{2}-1)}(\log_3 u)^{-2/3}$$

for $u \geq 20$, and $\mathcal{L}(u) = \mathcal{L}(20)$ else. By the definition (3.2) of $B(t)$, this implies that

$$-B(t) \leq C + \frac{\varepsilon_1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k (Xu)^{1/2} \mathcal{L}(Xu) du,$$

for all $t > 0$. Estimating this integral by Hafner's Lemma 2.3.6 in [3], we obtain

$$-B(t) \leq C + C_1 \varepsilon_1 (kX)^{1/2} \mathcal{L}(kX) = C + C_1 \varepsilon_1 t \mathcal{L}(t^2),$$

recalling (3.1). Together with (3.14), this yields a positive lower bound for ε_1 and thus completes the proof of our Theorem. \square

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